

UNLIMITED  
DISTRIBUTION  
ILLIMITÉE

5



DEFENCE RESEARCH ESTABLISHMENT  
ATLANTIC

AD A 124 948

DTIC FILE COPY

SOME SIMPLE FORMULAE FOR  
NORMAL MODE WAVE NUMBERS, CUTOFF FREQUENCIES,  
AND THE NUMBER OF MODES TRAPPED  
BY A SOUND CHANNEL

DTIC  
ELECTE  
FEB 25 1993  
S D D



D. R. E. A. TECHNICAL MEMORANDUM 82 / 0

RESEARCH AND DEVELOPMENT BRANCH  
DEPARTMENT OF NATIONAL DEFENCE  
CANADA

DEFENCE RESEARCH ESTABLISHMENT  
ATLANTIC

DARTMOUTH N.S.

D R E A TECHNICAL MEMORANDUM 82 / 0

SOME SIMPLE FORMULAE FOR  
NORMAL MODE WAVE NUMBERS, CUTOFF FREQUENCIES,  
AND THE NUMBER OF MODES TRAPPED  
BY A SOUND CHANNEL\*

Dale D. Ellis

November 1982

Approved by R. F. Brown

Director / Underwater Acoustics Division

DISTRIBUTION APPROVED BY



CHIEF D. R. E. A.

RESEARCH AND DEVELOPMENT BRANCH  
DEPARTMENT OF NATIONAL DEFENCE  
CANADA

\* Presented at the 101st meeting of the Acoustical Society of America in  
Ottawa, 18-22 May, 1981.

# ABSTRACT

To a good first approximation acoustic propagation in an underwater sound channel is dominated by a finite number of trapped modes. However, exact solutions are known for only a few special cases, making it necessary in general to use numerical methods to solve the normal mode equation. But often one is interested only in the gross features, such as the number of modes or cutoff frequencies, and one does not need the detail provided by a complete normal mode calculation. Even if a normal mode calculation is desired, the computation time can be reduced considerably if the mode wavenumbers can be estimated in advance. In such a case, the WKB method can be used to obtain formulae which, although they are approximate, are given in closed form. In this paper formulae based on exact and WKB solutions are presented for the number of modes trapped in some simple sound channels and for the wave numbers and cutoff frequencies associated with these modes. The number of trapped modes is shown to depend on the gross features of the sound channel, while the distribution of modal wave numbers depends to a greater degree on the details of the sound speed profile shape.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	



## RESUME

Dans une bonne première approximation, la propagation acoustique dans un canal de son sous-marin est dominée par un nombre fini de modes piégés. Cependant, des solutions exactes ne sont connues que pour quelques cas spéciaux, ce qui oblige en général à utiliser des méthodes numériques pour résoudre l'équation du mode normal. Souvent pourtant, le chercheur ne s'intéresse qu'aux caractéristiques brutes comme le nombre de modes ou les fréquences de coupure et il n'a pas besoin de la quantité de détails fournie par un calcul complet du mode normal. Même, lorsqu'un calcul du mode normal est voulu, le temps de calcul peut être considérablement réduit si les nombres d'onde du mode peuvent être estimés auparavant. Dans un tel cas, la méthode WKB peut servir à obtenir des formules qui, bien qu'elles soient approximatives, se présentent sous une forme fermée. Dans cette communication, l'auteur présente des formules basées sur des solutions exactes et sur des approximations WKB pour le nombre de modes piégés dans des canaux de son simple et pour les nombres d'onde et les fréquences de coupure liés à ces modes. Il démontre que le nombre de modes piégés dépend des caractéristiques brutes du canal de son, tandis que la distribution des nombres d'onde modaux dépend dans une plus grande mesure des détails de la forme du profil de la vitesse du son.

## TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	ii
TABLE OF CONTENTS	iv
1. INTRODUCTION	1
2. THE NORMAL MODE EQUATION	1
3. THE WKB METHOD	3
4. THE NUMBER OF TRAPPED MODES	5
5. CUTOFF FREQUENCIES	6
6. WAVE NUMBERS	6
7. EXAMPLE	7
8. DISCUSSION	7
ACKNOWLEDGEMENT	8
REFERENCES	8
DOCUMENT CONTROL DATA	9

## 1 Introduction

Acoustic propagation in an underwater sound channel is dominated by a finite number of trapped modes whose wavenumbers depend on the sound speed profile in the channel. Exact solutions are known for only a few special profiles, making it necessary in general to use numerical methods to solve the normal mode equation. But often one is interested only in the gross features, such as the number of modes or the cutoff frequencies, and one does not need the detail provided by a complete normal mode calculation. Moreover, even if a normal mode calculation is desired, the computation time can be reduced considerably if the mode numbers can be estimated in advance. In such cases the WKB (after Wentzel-Kramers-Brillouin and others) method can be used to obtain formulae which, although they are approximate<sup>1</sup>, are given in closed form.

In this paper, formulae based on WKB solutions are presented for the number of modes trapped in some simple sound channels and for the wave numbers and cutoff frequencies associated with these modes. The number of trapped modes is shown to depend on the gross features of the sound channel, while the distribution of mode wavenumbers depends to a greater degree on the details of the profile shape. Results are presented for the square (isovelocity-channel) profile, the parabolic profile, and the bilinear profile. An example shows how the simple formulae can be applied to a realistic ocean environment.

While the analysis is presented in terms of underwater acoustics, the results are applicable to other areas, such as transmission in an inhomogeneous waveguide or to the solution of the Schrodinger equation. Most of the results presented here have been known for some time<sup>2</sup>, but what is new is that the formulae for some of the more complicated waveguides can be put in the same functional form as the well known formulae for the ideal waveguide. The physical parameters, such as frequency, depth, and sound speed are easily distinguished from the details of the shape of the sound speed profile, which can be treated as a dimensionless quantity. The very simple form of the expressions makes them useful for back of the envelope calculations or for use with a pocket calculator. Moreover, using the same functional form for the expressions allows the effect of the shape of the sound speed profile to be easily seen.

## 2 The normal mode equation

The normal mode equation can be written as,

$$u_n''(z) + [\omega^2/c^2(z) - k_n^2]u_n(z) = 0 \quad (1)$$

where,

$z$  is the depth coordinate (increasing with depth from the surface),

$\omega = 2\pi f$  is the angular frequency,

$c(z)$  is the sound speed as a function of depth,

$k_n$  is the wave number or eigenvalue,

$u_n(z)$  is the normal mode function,

and  $u''$  denotes the second derivative of  $u$  with respect to  $z$ .

In general one wants to determine the normal mode wave numbers  $k_n$  and the associated mode functions  $u_n(z)$ , subject to certain boundary conditions. Notice the quantity  $[\omega^2/c^2(z) - k_n^2]$  which will be important in the discussion later; in particular, it is equal to zero at a turning point, where  $\omega/c(z) = k_n$ .

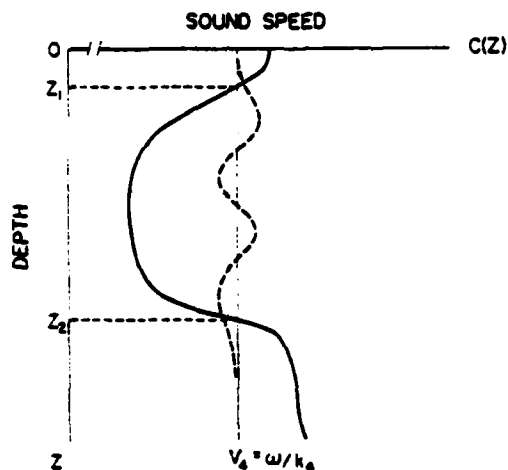


Figure 1. Normal mode solution (heavy dashed line) superimposed on a sound speed profile (heavy solid line).

Figure 1 shows a sound speed profile (heavy solid line) with a normal mode function (heavy dashed line) superimposed (with arbitrary amplitude) at the appropriate phase velocity  $v_n = \omega/k_n$ . Notice a number of things about the mode function:

- 1) At the turning points  $z_1$  and  $z_2$ ,  $\omega/c(z_1) = \omega/c(z_2) = k_n$ . These are the classical turning points for the equivalent ray travelling in the sound channel.
- 2) At the air-water interface the pressure, and hence the mode function, is zero; i.e.  $u_n(0) = 0$ .
- 3) Between 0 and  $z_1$  the normal mode function has an increasing exponential type of behaviour.
- 4) Between  $z_2$  and  $\infty$  the solution has a decreasing exponential type of behaviour.
- 5) Between  $z_1$  and  $z_2$  there are three zero crossings since this is the fourth mode; (in general the  $n$ -th mode will have  $n-1$  zero crossings).

Furthermore, note the sinusoidal behaviour between  $z_1$  and  $z_2$ ; note also that  $u_n \approx \sin(\pi/4)$  at the upper turning point and  $u_n \approx \sin(n-1/4)\pi$  at the lower turning point.

### 3 The WKB method

If the sound speed profile is changing slowly with respect to an acoustic wavelength, the WKB approximation<sup>1</sup> allows the solution to be written in terms of a slowly varying amplitude  $r(z)$  and a monotonically increasing phase  $\phi(z)$ :

$$u_n(z) = M r(z) \sin[\phi(z)] \quad (2)$$

where

$$r(z) = [\omega^2/c^2(z) - k_n^2]^{-1/4} \quad (3)$$

$$\phi(z) = \int_{z_1}^z [\omega^2/c^2(z') - k_n^2]^{1/2} dz' + \delta_1, \quad z_1 < z < z_2 \quad (4)$$

and  $M$  is a normalization constant. The integral in Eq. (4) will be referred to as the phase integral. Note several points:

- 1) the term in square brackets is the same term that appeared in Eq. (1).
- 2) near a turning point  $r(z)$  is singular.
- 3)  $\phi(z)$  is well behaved, however, and can be used to determine the WKB eigenvalues  $k_n$ .
- 4)  $\phi(z_1) = \delta_1$  at the upper turning point.
- 5)  $\phi(z_2) = n\pi - \delta_2$  at the lower turning point.

Figure 2 shows how the phase  $\delta_1$  at the upper turning point depends on boundary effects. If the surface is a pressure release one, the pressure is zero,  $u_n(0) = 0$  and the phase  $\delta_1 = 0$ ; if the surface is rigid, the normal derivative of the pressure is zero, i.e.  $u_n'(0) = 0$ , and the phase  $\delta_1 = \pi/2$ . At a turning point the phase  $\delta_1$  is between 0 and  $\pi/2$ ;  $\delta_1 = \pi/4$  is the usual choice<sup>1</sup>. The same comments apply to the phase at the lower turning point.

In the WKB method the total phase change between the turning points is given by:

$$\int_{z_1}^{z_2} [\omega^2/c^2(z) - k_n^2]^{1/2} dz + \delta_1 + \delta_2 = n\pi. \quad (5)$$



PRESSURE RELEASE

RIGID

TURNING POINT

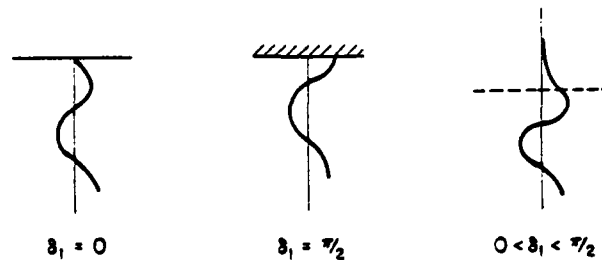


Figure 2. Effect of various boundary types on the phase  $\delta$ .

where  $\delta_1$  and  $\delta_2$  are the phases at the upper and lower turning points. With the usual choice of  $\delta_1 = \delta_2 = \pi/4$  the WKB eigenvalue equation becomes

$$\int_{z_1}^{z_2} [\omega^2/c^2(z) - k_n^2]^{1/2} dz = (n-1/2)\pi \quad (6)$$

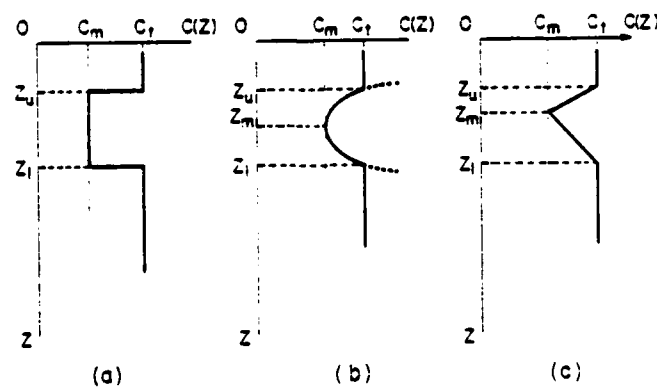


Figure 3. Three simple sound channel shapes: (a) the square isovelocity-channel profile, (b) the parabolic sound channel, and (c) the bilinear sound channel.

One wishes to solve equation (6) for  $k_n$  which appears explicitly in the integrand and implicitly in the limits  $z_1$  and  $z_2$ . However, for certain sound speed profiles  $c(z)$  the integral can be evaluated analytically and an expression obtained for  $k_n$ . Three such sound speed profiles are shown in Figure 3 : (a) the square isovelocity-channel profile, (b) the parabolic sound channel, and (c) the bilinear sound channel. Note that it is actually  $c^{-2}$  rather than  $c$  which is parabolic or linear. Some notation is also introduced at this point:  $c_m$  = the minimum sound speed in the channel,  $c_t$  = the maximum sound speed in the channel, and  $h = z_t - z_b$  = the maximum vertical extent of the sound channel.

#### 4 The number of trapped modes

The WKB eigenvalue equation can be solved for the number of trapped modes  $N$  if the phase integral can be evaluated either analytically or numerically. Using  $n \rightarrow N$ ,  $z_1 = z_b$ ,  $z_2 = z_t$ ,  $k_N = \omega/c$ , and rearranging Eq. (6) gives  $h = z_t - z_b$  and  $x = z/h$ :

$$N = 1/2 + (2hf/c_m)[1 - c_m^2/c_t^2]^{1/2} \int_0^1 \{ [c_m^2/c^2(x) - c_m^2/c_t^2] / [1 - c_m^2/c_t^2] \}^{1/2} dx \quad (7)$$

By introducing the dimensionless quantities

$$a = [1 - c_m^2/c_t^2]^{1/2} \quad (8)$$

and

$$\alpha = \int_0^1 \{ [c_m^2/c^2(x) - c_m^2/c_t^2] / [1 - c_m^2/c_t^2] \}^{1/2} dx \quad (9)$$

equation (7) can then be written as

$$N = 1/2 + (2hf/c_m) a \alpha \quad (10)$$

The quantity  $a$  is introduced strictly for notational convenience. The quantity  $\alpha$ , however, is related to the shape of the sound speed profile, but contains none of the physical parameters such as the frequency, depth or sound speeds. Note that in the case of an isovelocity or square profile  $\alpha = 1$ , and Eq. (10) gives the classical formula for the number of trapped modes. Figure 4 graphically illustrates the significance of the integral in Eq. (9), where the vertical extent of the channel  $z_t - z_b$  gets mapped into the range 0 to 1, and where the sound speeds between  $c_m$  and  $c_t$  get mapped into 0 to 1 and where the integrand of Eq. (9) (denoted by  $g(x)$ ) is enclosed in a square box of unit size. The integral  $\alpha$  is given by the shaded area, which can be calculated analytically or numerically or even estimated by eye.

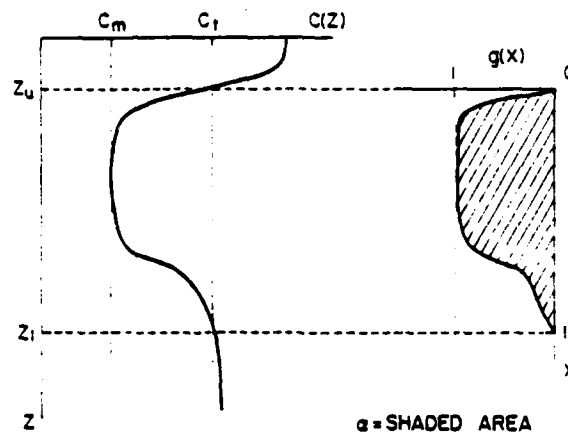


Figure 4. The sound channel and the associated function  $g(x)$ ; the shaded area  $\alpha$  is defined in Eq. (9).

### 5 Cutoff frequencies

The cutoff frequency for the  $n$ -th mode can be obtained by rearranging Eq. (10) to give

$$f_n^{\text{cut}} = [c_m(n-1/2)] / [2ha\alpha] \quad (11)$$

### 6 Wave numbers

For the three sound speed profiles shown in Figure 3, the eigenvalue equation (6) can be solved for the wavenumbers  $k_n$ , giving results of the form

$$k_n^2 = \omega^2/c_m^2 - A(n-1/2)^p \quad (12)$$

where the specific values of  $A$  and  $p$  are given in Table 1. For the three profiles considered  $p$  varies between  $2/3$  and  $2$ , while the corresponding values of the shape parameter  $\alpha$  varies only between  $2/3$  and  $1$ . Moreover,  $\alpha$  enters equations (10) and (11) in a linear fashion, while  $p$  appears as an exponent in equation (12). Thus, the wave numbers are much more sensitive to the profile shape than are the trapped modes and the cutoff frequencies.

Table 1

Specific values of  $\alpha$ ,  $p$  and  $A$  for three simple profiles.

Profile Shape	$\alpha$	$p$	$A$
Square	1	2	$(\pi/h)^2$
Parabolic	$\pi/4 = 0.79$	1	$4a\omega/(c_m h)$
Bilinear	2/3	2/3	$[3\pi\omega^2 a^2/(2c_m^2 h)]^{2/3}$

## 7 Example

Table 2 shows an example of how the formulae might be applied to a realistic ocean environment, and compares the wave numbers with those obtained from a normal mode calculation. The sound speed profile approximates a typical summer sound speed profile in 100 m of water on the Scotian Shelf: a 20 m isovelocity layer of speed 1520 m/s at the surface, a minimum sound speed of 1460 m/s at a depth of 40 m, and a speed of 1490 m/s at the bottom. The table compares the wave numbers, or phase velocities, obtained using the bilinear formula with those from a complete normal mode calculation at 200 Hz.

Table 2

Comparison of the phase velocities obtained from equation (12)

with those from a complete normal mode calculation.

Mode Number	Phase Velocities (m/s)		
	Normal mode	Equation (12)	Difference
1	1465.72	1465.46	-0.26
2	1471.34	1471.43	0.09
3	1475.99	1476.15	0.16
4	1480.13	1480.29	0.16
5	1484.94	1484.09	0.15
6	1487.48	1487.63	0.15

The WKB method gives a good approximation to the normal mode wave numbers; four or five digits accuracy as in the above example is not unreasonable. In fact for the parabolic profile the WKB and exact calculations give the identical results. Provided that this is of sufficient accuracy, the value of the analytic formula is obvious from a computational point of view.

## 8 Discussion

Equations (10)-(12) together with the values of  $\alpha$ ,  $p$  and  $A$  given in Table 1 summarize the results of this paper: simple analytical formulae for normal mode wave numbers, cutoff frequencies, and the number of modes trapped in a sound channel of simple shape. The factor of 1/2 appearing in Eqs. (10)-(12) can be generalized to a  $\delta$  which depends on the boundaries of the sound channel as well as the type of turning point that the mode "sees". Equations (10) and (11) for the number

of modes and the cutoff frequencies are useful for more general profiles provided that the integral of Eq. (9) can be estimated.

The results show the sensitivity of the modes to the shape of the sound speed profile: the number of modes depends on the shape parameter  $\alpha$  of the sound speed profile, while the distribution of mode numbers is more sensitive to the details of the profile shape.

#### ACKNOWLEDGEMENT

Useful discussions were held with my colleagues at D.R.E.A., in particular with David Chapman who made a number of valuable suggestions.

#### REFERENCES

1. L. I. Schiff, Quantum Mechanics, McGraw Hill, 1968.
2. I. Tolstoy and C. S. Clay, Ocean Acoustics, McGraw Hill, 1968.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)		
1. ORIGINATING ACTIVITY DEFENCE RESEARCH ESTABLISHMENT ATLANTIC		2a. DOCUMENT SECURITY CLASSIFICATION UNCLASSIFIED
		2b. GROUP
3. DOCUMENT TITLE SOME SIMPLE FORMULAE FOR NORMAL MODE WAVE NUMBERS, CUTOFF FREQUENCIES, AND THE NUMBER OF MODES TRAPPED BY A SOUND CHANNEL		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) JOURNAL PUBLICATION & TECH MEMO		
5. AUTHOR(S) (Last name, first name, middle initial) ELLIS, DALE D.		
6. DOCUMENT DATE NOVEMBER 1982	7a. TOTAL NO. OF PAGES 14	7b. NO. OF REFS 2
8a. PROJECT OR GRANT NO.	9a. ORIGINATOR'S DOCUMENT NUMBER(S) DREA TECHNICAL MEMORANDUM 82/O	
8b. CONTRACT NO.	9b. OTHER DOCUMENT NO.(S) (Any other numbers that may be assigned this document)	
10. DISTRIBUTION STATEMENT		
11. SUPPLEMENTARY NOTES		12. SPONSORING ACTIVITY
13. ABSTRACT To a good first approximation acoustic propagation in an underwater sound channel is dominated by a finite number of trapped modes. However, exact solutions are known for only a few special cases, making it necessary in general to use numerical methods to solve the normal mode equation. But often one is interested only in the gross features, such as the number of modes or cutoff frequencies, and one does not need the detail provided by a complete normal mode calculation. Even if a normal mode calculation is desired, the computation time can be reduced considerably if the mode wavenumbers can be estimated in advance. In such a case, the WKB method can be used to obtain formulae which, although they are approximate, are given in closed form. In this paper formulae based on exact and WKB solutions are presented for the number of modes trapped in some simple sound channels and for the wave numbers and cutoff frequencies associated with these modes. The number of trapped modes is shown to depend on the gross features of the sound channel, while the distribution of modal wave numbers depends to a greater degree on the details of the sound speed profile shape.		

1/513  
70-716

## KEY WORDS

UNDERWATER ACOUSTICS  
 NORMAL MODES  
 SOUND CHANNEL  
 WAVE GUIDES  
 WKB

## INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the organization issuing the document.
- 2a. **DOCUMENT SECURITY CLASSIFICATION:** Enter the overall security classification of the document including special warning terms whenever applicable.
- 2b. **GROUP:** Enter security reclassification group number. The three groups are defined in Appendix M of the DRB Security Regulations.
3. **DOCUMENT TITLE:** Enter the complete document title in all capital letters. Titles in all cases should be unclassified. If a sufficiently descriptive title cannot be selected without classification, show title classification with the usual one-capital-letter abbreviation in parentheses immediately following the title.
4. **DESCRIPTIVE NOTES:** Enter the category of document, e.g. technical report, technical note or technical letter. If appropriate, enter the type of document, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.
5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the document. Enter last name, first name, middle initial. If military, show rank. The name of the principal author is an absolute minimum requirement.
6. **DOCUMENT DATE:** Enter the date (month, year) of Establishment approval for publication of the document.
- 7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.
- 7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the document.
- 8a. **PROJECT OR GRANT NUMBER:** If appropriate, enter the applicable research and development project or grant number under which the document was written.
- 8b. **CONTRACT NUMBER:** If appropriate, enter the applicable number under which the document was written.
- 9a. **ORIGINATOR'S DOCUMENT NUMBER(S):** Enter the official document number by which the document will be identified and controlled by the originating activity. This number must be unique to this document.
- 9b. **OTHER DOCUMENT NUMBER(S):** If the document has been assigned any other document numbers (either by the originator or by the sponsor), also enter this number(s).
10. **DISTRIBUTION STATEMENT:** Enter any limitations on further dissemination of the document, other than those imposed by security classification, using standard statements such as:
  - (1) "Qualified requesters may obtain copies of this document from their defence documentation center."
  - (2) "Announcement and dissemination of this document is not authorized without prior approval from originating activity."
11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.
12. **SPONSORING ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring the research and development. Include address.
13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document, even though it may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall end with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (TS), (S), (C), (R), or (U).  
  
The length of the abstract should be limited to 20 single-spaced standard typewritten lines; 7 1/4 inches long.
14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a document and could be helpful in cataloging the document. Key words should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context.